$$\lambda = \frac{\max \text{Prob}(\text{data} \mid H_0)}{\max \text{Prob}(\text{data} \mid H_1)}$$

H_0: $\text{Prob}(\text{data} \mid H_0) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i - \alpha)^2} = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\Sigma(y_i-\alpha)^2}$

is maximized at $\begin{cases} \hat{\alpha}_{MLE} = \bar{y} \\ \hat{\sigma}^2_{MLE} = \hat{\sigma}_0^2 = \frac{1}{n}\Sigma(y_i - \bar{y})^2 \end{cases}$ to be $(2\pi\hat{\sigma}_0^2)^{-\frac{n}{2}} e^{-\frac{n}{2}}$

or $(2\pi e \hat{\sigma}_0^2)^{-\frac{n}{2}}$.

H_1: $\text{Prob}(\text{data} \mid H_1) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(y_i - \alpha - \beta x_i)^2} = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\Sigma(y_i - \alpha - \beta x_i)^2}$

is maximized at $\left\{\begin{aligned} \hat{\beta} &= \frac{\Sigma_i (x_i - \bar{x})(y_i - \bar{y})}{\Sigma_i (x_i - \bar{x})^2} \\ \hat{\alpha} &= \bar{y} - \hat{\beta}\bar{x} \\ \hat{\sigma}_1^2 &= \frac{1}{n}\Sigma_i (y_i - \hat{\alpha} - \hat{\beta}x_i)^2 \end{aligned}\right\}$ to be

$\max \text{Prob}(\text{data} \mid H_1) = (2\pi e \hat{\sigma}_1^2)^{-\frac{n}{2}}$.

Therefore, $\lambda = \left(\frac{2\pi e \hat{\sigma}_1^2}{2\pi e \hat{\sigma}_0^2}\right)^{\frac{n}{2}} = \left(\frac{\hat{\sigma}_1^2}{\hat{\sigma}_0^2}\right)^{\frac{n}{2}} = \left[\frac{\Sigma_i (y_i - \hat{\alpha} - \hat{\beta}x_i)^2}{\Sigma(y_i - \bar{y})^2}\right]^{\frac{n}{2}}$.

Numerator is $\Sigma[y_i - \bar{y} - \hat{\beta}(x_i - \bar{x})]^2$, since $\bar{y} = \hat{\beta}\bar{x} + \hat{\alpha}$. ← "Linear Regression".

Denominator is $\Sigma[\underbrace{y_i - \bar{y} - \hat{\beta}(x_i - \bar{x})}_{A} + \underbrace{\hat{\beta}(x_i - \bar{x})}_{B}]^2$ by inventing terms.

$(A + B)^2$

$\to \Sigma\{[y_i - \bar{y} - \hat{\beta}(x_i - \bar{x})]^2 + 2[y_i - \bar{y} - \hat{\beta}(x_i - \bar{x})]\cdot\hat{\beta}(x_i - \bar{x}) + [\hat{\beta}(x_i - \bar{x})]^2\}$

Since $\hat{\beta} = \frac{\Sigma(x_i - \bar{x})(y_i - \bar{y})}{\Sigma(x_i - \bar{x})^2}$, this middle term is zero.

so $\lambda = \left[\frac{\Sigma[y_i - \bar{y} - \hat{\beta}(x_i - \bar{x})]^2}{\Sigma[y_i - \bar{y} - \hat{\beta}(x_i - \bar{x})]^2 + \Sigma[\hat{\beta}(x_i - \bar{x})]^2}\right]$ to the $\frac{n}{2}$-th order.

$= \left[\frac{1}{1 + \frac{\Sigma[\hat{\beta}(x_i - \bar{x})]^2}{\Sigma[y_i - \bar{y} - \hat{\beta}(x_i - \bar{x})]^2}}\right]^{\frac{n}{2}}$. By setting $t^2 := \frac{\hat{\beta}\Sigma(x_i - \bar{x})}{\frac{\Sigma[y_i - \bar{y} - \hat{\beta}(x_i - \bar{x})]^2}{n-2}}$.

$\lambda = \left(\frac{1}{1 + \frac{t^2}{n-2}}\right)^{\frac{n}{2}}$. $t \sim T(\nu = n-2)$.

If t is large enough (according to t-chart), we reject H_0.